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The Singularity $x\partial/\partial y$

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1. Introduction

In [2], F. Takens gives the classification of codimension two singularities of vector fields (i.e. those singularities which generically occur in two parameter families of vector fields) in any dimension.

F. Dumortier considers the special case of vector fields in the plane in [1] and proves a stability theorem which enables him to classify the codimension four singularities of vector fields with separatrices. Also it is conjectured in [1] that the singularities of vector fields with separatrices can be classified for all codimensions. It is this conjecture which is considered here.

2. Definitions

An exhaustive set of definitions is given in both [1] and [2]. Here we give only the basic definitions.

Let X be a C^∞-vectorfield on the plane $\mathbb{R}^2$. We are going to study the topological behaviour of the vectorfield X in the neighbourhood of a point 0, (assumed to be the origin), where $X(0) = 0$. We say X has a singularity in 0. The problem is of local type and so we work in G^2, the space of germs of C^∞-vectorfields vanishing at the origin.

Definition 2.1. Let $\tilde{X}, \tilde{X}' \in G^2$. Then $\tilde{X}, \tilde{X}'$ are k-jet equivalent if for some representatives X, X' and some admissable coordinate system all the partial derivatives up to, and including, order k of the component functions of X in 0 coincide with those of X'. The equivalence classes are called k-jets; the k-jet of X being denoted by $j_k(X)$.

There is a natural 1-1 correspondence between k-jets of vector fields on $\mathbb{R}^2$ vanishing at the origin and vector fields with component functions which are polynomials of degree $\leqslant k$. The set J_k of k-jets of singularities of vector fields on $\mathbb{R}^2$ is a vector space and hence has a natural manifold structure. The topology

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on G^2 is the coarsest topology for which the natural projections $\pi_k: G^2 \to J_k$ are continuous, where $\pi_k(\tilde{X}) = j_k(X)$ for some representative X of the germ $\tilde{X}$.

DEFINITION 2.2. We call a subset W_k' of J_k semi-algebraic, if it is the finite union of a number of sets which can be defined by polynomial equalities and inequalities. A semi-algebraic subset $W \subseteq G^2$ is of the form $W = \pi_k^{-1}(W_k')$ where W_k' is a semi-algebraic subset of J_k.

DEFINITION 2.3. The germs $\tilde{X}, \tilde{X}' \in G^2$ are topologically (or C^0-) equivalent if for some representatives X, X' there exist neighbourhoods U, V of 0 in $\mathbb{R}^2$ and a homeomorphism $h : U \to V$ mapping integral curves of X onto integral curves of X' preserving the sense but not necessarily the parametrization.

DEFINITION 2.4. Let $K \subseteq G^2$ and $X \in K$. Then we say that X is $K - C^0$-stable if there is a neighbourhood U of X in G^2 such that every $X' \in U \cap K$ is C^0-equivalent with X.

DEFINITION 2.5. A vectorfield X on $\mathbb{R}^2$, $X(0) = 0$, has a separatrix in 0, if for some neighbourhood V of 0 there exists an integral curve γ of X through $P \in V$ with natural parametrization $t \mapsto \gamma(t) = (x(t), y(t)) \in \mathbb{R}^2$, $(\gamma(0) = P)$, remaining in V for $t \geqslant 0$ (resp. $t \leqslant 0$) such that

— $\|\gamma(t)\| > 0$, for all $t \geqslant 0$ (resp. $t \leqslant 0$).

— $\gamma(t) \to 0$ as $t \to \infty$ (resp. $t \to -\infty$).

— $\exists$ a continuous function $\theta(t)$ with $\tan(\theta(t)) = y(t)/x(t)$ such that $\theta(t)$ tends to a finite limit θ_0 as $t \to \infty$ (resp. $t \to -\infty$).

3. STATEMENT OF THE RESULTS

THEOREM 1. *There is an infinite sequence of closed semi-algebraic subsets* $G^2 \supset V_2 \supset V_3 \supset \cdots \supset V_j \supset \cdots$, *where* V_j *is of codimension* j *and* $V_2 = \{\tilde{X} \mid j_1(\tilde{X})(0) = x\, \partial/\partial y\}$, *such that for each* $j \geqslant 2$, $V_j \backslash V_{j+1}$ *is a finite union of non-singular open codimension j-manifolds* M_{ij} *where* $V_j \backslash V_{j+1} = \bigcup_{i=1}^{n_j} M_{ij}$. *Let* $\tilde{R} \subset V_2$ *be the set of germs of vectorfields without separatrices. Then each* $X \in (V_2 \backslash \tilde{R}) \cap M_{ij}$ *is* $M_{ij} - C^0$*-stable.*

THEOREM 2. *There are only seven singularities appearing in the classification up to local* C^0*-equivalence.*

Remarks. (i) By the "Takens normal form" theorem [2] the infinite jets of vectorfields in V_2 are given by

$$j_\infty(X)(0) = \left(x + \sum_{i=2}^{\infty} b_i y^i\right) \frac{\partial}{\partial y} + \left(\sum_{j=2}^{\infty} a_j y^j\right) \frac{\partial}{\partial x} \tag{1}$$

(ii) We show that $\tilde{R}$ is the set of germs in 0 of vectorfields with the normal form (1) where $b_2^2 + 2a_3 < 0$, $a_2 = 0$, or $b_{n+1}^2 + [4/(n+1)]\, a_{2n+1} < 0$ and $b_2 = \cdots = b_n = a_2 = \cdots = a_{2n} = 0$ for general n. This is proved in Theorem 1.

(iii) Using the "Jordan normal form" we can already subdivide J_1 and hence G^2 into 5-semi-algebraic sets.

$$W_1' = \{X \mid \Delta(j_1(X)(0)) \neq 0\}$$
$$W_2' = \{X \mid \Delta(j_1(X)(0)) = 0,\ T(j_1(X)(0)) \neq 0\}$$
$$W_3' = \{X \mid \Delta(j_1(X)(0)) > 0,\ T(j_1(X)(0)) = 0\}$$
$$W_4' \cup W_5' = \{X \mid \Delta(j_1(X)(0)) = 0,\ T(j_1(X)(0)) = 0\}$$
$$W_5' = \{X \mid j_1(X)(0) \equiv 0\},$$

where $\Delta(j_1(X)(0))$ and $T(j_1(X)(0))$ are the determinant and trace of the matrix of the first partial derivatives of X in 0.

It is clear from the theorem of Seidenberg–Tarski [2] that the algebraic properties of the W_j' also hold for the W_j.

By expressing the vectorfields in W_2' and W_3' in Takens normal form it is easy to classify the vectorfields for all codimensions (see [1]).

In Section 5 we classify the singularities occurring in W_4'. To completely answer the conjecture it remains to classify the singularities occurring where the vector fields have zero 1-jet which are not amenable to study by normal form techniques.

The technique we use to investigate W_4' is described in Section 4.

4. The Blowing-Up Construction for Vector Fields

Let X be a C^∞-vectorfield on $\mathbb{R}^2$, or in a neighbourhood of 0 with $X(0) = 0$ and $\Phi\colon S^1 \times \mathbb{R} \to \mathbb{R}^2$ be the map defining polar coordinates, $\Phi(\alpha, r) = (r\cos\alpha, r\sin\alpha)$. Then there exists a C^∞-vectorfield $\tilde{X}$ on $S^1 \times \mathbb{R}$ such that for each $q \in S^1 \times \mathbb{R}$, $\Phi_*(\tilde{X}(q)) = X(\Phi(q))$. Furthermore if $j_k(X)(0) \equiv 0$ and $j_{k+1}(X)(0) \not\equiv 0$ then we define $\bar{X}(q) = (1/r^k)\,\tilde{X}(q)$.

We say that $\bar{X}$ is the polar blow-up of X in 0. Clearly $S^1 \times \{0\} = \Phi^{-1}(0)$ is $\bar{X}$-invariant and we let $\bar{X}_0 = \eta_1(\alpha, 0)\,\partial/\partial\alpha$, $\alpha \in S^1$ be the restriction of $\bar{X}$ to $S^1 \times \{0\}$. Each singularity of $\bar{X}_0$ can now be considered in turn and possibly be itself blown up.

F. Dumortier [1] proves that if after sufficiently many blowing ups the singularities which one obtains are each hyperbolic or partially hyperbolic, then the singularity scheme obtained determines the topological type of X in a neighbourhood of 0 providing X has at least one separatrix. Moreover if a set K of such vectorfields gives rise to the same singularity schemes then any vectorfield in the set is $K - C^0$-stable. For further details on how blow ups are used to determine topological type, see [2, p. 75].

Notation. If X is given as above and $\eta_1(\alpha, 0) = 0$ at $\alpha = \alpha_0$ say then we define $\bar{X}_{[\alpha_0]}$ to be $\bar{X}$ in a neighbourhood of $(\alpha_0, 0) \in S^1 \times \mathbb{R}$.

The special cases $\bar{X}_{[0]}$, $\bar{X}_{[\pi/2]}$ are called the blow-ups in the x and y-directions of X and can be easily calculated by using the following formulae,

$$\begin{aligned}\bar{X}_{[0]} = {} & ((1/x')^k)(X_1(x', x'y'))\, \partial/\partial x' \\ & + ((1/x')^{k+1})(X_2(x', x'y') - y'X_1(x', x'y'))\, \partial/\partial y',\end{aligned}$$

and

$$\begin{aligned}\bar{X}_{[\pi/2]} = {} & ((1/y')^{k+1})(X_1(x'y', y') - x'X_2(x'y', y'))\, \partial/\partial x' \\ & + ((1/y')^k)\, X_2(x'y', y')\, \partial/\partial y'\end{aligned}$$

which are induced from $X = X_1(x, y)\, \partial/\partial x + X_2(x, y)\, \partial/\partial y$ by the changes of coordinates $(x', y') \mapsto (x, y) = (x', x'y')$ and $(x', y') \mapsto (x'y', y')$ respectively. The integer k is defined such that $j_k(X)(0) \equiv 0$ and $j_{k+1}(X)(0) \not\equiv 0$.

5. The Proofs of Theorems 1 and 2

We use the Takens normal form for vectorfields $X \in W_4'$, that is

$$j_\infty(X)(0) = \left(x + \sum_{i=2}^{\infty} b_i y^i\right) \frac{\partial}{\partial y} + \left(\sum_{j=2}^{\infty} a_j y^j\right) \frac{\partial}{\partial x}.$$

There are two special classes of subsets of W_4' that occur repeatedly in the analysis

$$S_{ij} = \{X \in W_4' \mid a_2 = \cdots = a_i = 0,\ b_2 = \cdots = b_{j-1} = 0,\ b_j \neq 0\}$$

$$T_{ij} = \{X \in W_4' \mid a_2 = \cdots = a_{i-1} = 0,\ a_i \neq 0,\ b_2 = \cdots = b_{j-1} = 0,\ b_j \neq 0\}$$

For $n \in \mathbb{N}$ define the sets $W_{4n}' \subseteq W_4'$ as follows

$$W_{42}' = W_4'$$

$$W_{43}' = \{X \in W_4' \mid a_2 = 0\}$$

$$W_{44}' = \{X \in W_4' \mid a_2 = a_3 = 0\} \cup \{X \in W_4' \mid a_2 = 0,\ a_3 \neq 0,\ b_2^2 + 2a_3 = 0\}$$

and in general for any integer $n \geqslant 1$

$$W'_{4,3n} = \{X \in W'_4 \mid a_2 = \cdots = a_{2n} = 0,\ b_2 = \cdots b_n = 0\} \cup \left\{\bigcup_{j=2}^{n-1} S_{3n+1-j,j}\right\}$$

$$W'_{4,3n+1} = \{X \in W'_4 \mid a_2 = \cdots = a_{2n+1} = 0,\ b_2 = \cdots = b_n = 0\}$$
$$\cup\ \{X \in W'_4 \mid a_2 = \cdots = a_{2n} = 0,\ a_{2n+1} \neq 0,\ b_2 = \cdots = b_n = 0,$$
$$\times\ b_{n+1}^2 + \frac{4}{n+1}\, a_{2n+1} = 0\} \cup \left\{\bigcup_{j=2}^{n} S_{3n+2-j,j}\right\}$$

$$W'_{4,3n+2} = \{X \in W'_4 \mid a_2 = \cdots = a_{2n+1} = 0,\ b_2 = \cdots = b_{n+1} = 0\}$$
$$\cup \left\{\bigcup_{j=2}^{n+1} S_{3n+3-j,j}\right\}.$$

Remark. The sets $W'_{42}\backslash W'_{43}$, $W'_{43}\backslash W'_{44}$ and $W'_{44}\backslash W'_{45}$ have already been considered in [1]. We can investigate all the other difference sets by consideration of the sets $W'_{4,3n-1}\backslash W'_{4,3n}$, $W'_{4,3n}\backslash W'_{4,3n+1}$ and $W'_{4,3n+1}\backslash W'_{4,3n+2}$. We find that the vector fields in these three sets have topological properties essentially independent of the integer n.

I. $W'_{4,3n-1}\backslash W'_{4,3n}$

$$= \{X \in W'_4 \mid a_2 = \cdots = a_{2n-1} = 0,\ a_{2n} \neq 0,\ b_2 = \cdots = b_n = 0\}$$
$$\cup \left\{\bigcup_{j=2}^{n} T_{3n+1-j,j}\right\}$$

(i) $\{X \in W'_4 \mid a_2 = \cdots = a_{2n-1} = 0,\ a_{2n} \neq 0,\ b_2 = \cdots = b_n = 0\}$

Let $\bar{X}$ be the polar blowing up of X. Then $\bar{X} \mid S^1 \times \{0\}$ vanishes at $\alpha = \pi/2$, $(3\pi/2)$, and

$$X_1 = X_{[\pi/2]} = \left(xy + \sum_{i \geqslant n+1} b_i y^i\right)\frac{\partial}{\partial y}$$
$$+ \left(-x^2 - \sum_{i \geqslant n+1} b_i x y^{i-1} + \sum_{j \geqslant 2n} a_j y^{j-1}\right)\frac{\partial}{\partial x}.$$

Blowing up X_1 in its own origin we get $\bar{X}_1 \mid S^1 \times \{0\} = 2\cos^2\alpha \sin\alpha\, \partial/\partial\alpha$. In $\alpha = 0$ $j_1(X_{1_{[0]}})(0) = -r\,\partial/\partial r + 2\alpha\,\partial/\partial\alpha$ which is hyperbolic. To analyse the singularity $\alpha = \pi/2$ we need to blow up X_1 in the y-direction a further $n - 1$ times to obtain

$$X_n = \left(xy + \sum_{i \geqslant n+1} b_i y^{i+1-n}\right)\frac{\partial}{\partial y}$$
$$+ \left(-nx^2 - n\left(\sum_{i \geqslant n+1} b_i x y^{i-n}\right) + \sum_{j \geqslant 2n} a_j y^{j+1-2n}\right)\frac{\partial}{\partial x}.$$

A polar blowing up of X_n reveals a singularity at $\alpha = 0, (\pi)$ and $j_2(X_{n_{[0]}})(0) = (a_{2n}xy - nx^2)\,\partial/\partial y + ((n+1)\,xy - a_{2n}y^2)\,\partial/\partial x$. A final blowing up X_{n+1} of X_{n+1} $(=X_{n_{[0]}})$ has singularities in $\alpha = 0, (\pi)$, $\alpha = \pi/2, (3\pi/2)$ and $\tan\alpha_0 = 2n + 1/2a_{2n}$.

$$\text{In}\quad \alpha = 0 : j_1(X_{n+1_{[0]}})(0) = (2n+1)\,\alpha\frac{\partial}{\partial\alpha} - (n-1)r\frac{\partial}{\partial r}$$

$$\alpha = \pi/2 : j_1(X_{n+1_{[\pi/2]}})(0) = 2a_{2n}\,\alpha\frac{\partial}{\partial\alpha} - a_{2n}r\frac{\partial}{\partial r}$$

$$\tan\alpha_0 = 2n + 1/2a_{2n} : j_1(X_{n+1_{[\alpha_0]}})(0) = -2a_{2n}(2n+1)^3\,\alpha\frac{\partial}{\partial\alpha} + (4a_{2n}^3 + (2n+1)^2 a_{2n})r\frac{\partial}{\partial r}.$$

If $a_{2n} < 0$ then this is topologically equivalent to the case $a_{2n} > 0$ by taking a reflection of coordinates in the origin (Fig. 1).

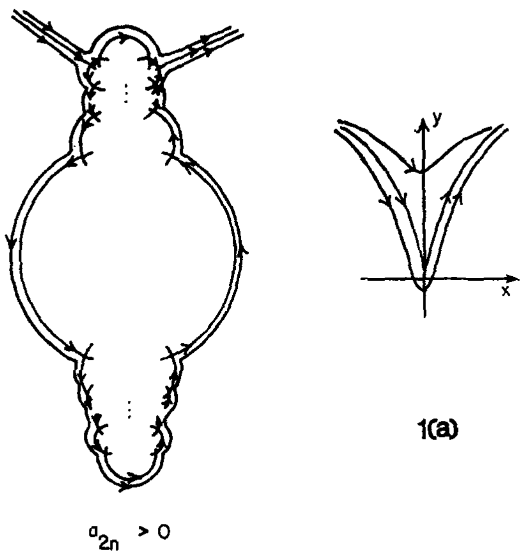


FIG. 1. A typical representative of vectorfields of type 1 is given by $X = x\,\partial/\partial y + y^2\,\partial/\partial x$ in 1a.

(ii) The vectorfields of type Tij, $2j \leqslant i$. The vectorfield $X \in Tij$, $2j \leqslant i$ has the normal form

$$X = \Big(x + \sum_{k \geqslant j} b_k y^k\Big)\frac{\partial}{\partial y} + \Big(\sum_{k \geqslant i} a_k y^k\Big)\frac{\partial}{\partial x}.$$

To investigate the singularity obtained in the polar blowing up of X in $\alpha = \pi/2(3\pi/2)$ we blow up X in the y-direction to obtain X_1.

$$X_1 = \left(xy + \sum_{k\geqslant j} b_k y^k\right)\frac{\partial}{\partial y} + \left(\sum_{k\geqslant i} a_k y^{k-1} - x\sum_{k\geqslant j} b_k y^{k-1}\right)\frac{\partial}{\partial x}.$$

A polar blowing up of X_1 reveals singularities of $X_1 \mid S^1 \times \{0\}$ at $\alpha = 0, (\pi)$ and $\alpha = \pi/2(3\pi/2)$ such that $j_1(X_{1_{[0]}})(0) = -r\,\partial/\partial r + 2\alpha\,\partial/\partial\alpha$ and $j_1(X_{1_{[\pi/2]}})(0) = 0$. A further $j-2$ blow ups in the y-direction gives the vectorfield X_{j-1} where

$$X_{j-1} = \left(xy + \sum_{k\geqslant j} b_k y^{k-j+2}\right)\frac{\partial}{\partial y} + \left(y^{-2j+3}\left(\sum_{k\geqslant i} a_k y^k\right)\right.$$
$$\left. -(j-1)x\left(x + \sum_{k\geqslant j} b_k y^{k+1-j}\right)\right)\frac{\partial}{\partial x}.$$

The polar blow up $\bar{X}_{j-1}$ is such that $\bar{X}_{j-1} \mid S^1 \times \{0\} = j\cos\alpha\sin\alpha(\cos\alpha + b_j\sin\alpha)$. If j is even $b_j = 1$ can be obtained by a linear change of coordinates and if j is odd we can similarly obtain $b_j = 1$ if $b_j > 0$ and $b_j = -1$ if $b_j < 0$.

(a) $b_j = +1$

$$j_1(X_{j-1_{[0]}})(0) = j\alpha\frac{\partial}{\partial\alpha} - (j-1)r\frac{\partial}{\partial r}$$

$$j_1(X_{j-1_{[\pi/2]}})(0) = -j\alpha\frac{\partial}{\partial\alpha} + r\frac{\partial}{\partial r}$$

and

$$j_1(X_{j-1_{[3\pi/4]}})(0) = \frac{j}{2^{1/2}}\alpha\frac{\partial}{\partial\alpha}.$$

Note that the singularity in $\alpha = 3\pi/4$ is partially hyperbolic. A calculation of the higher order jets in this singularity reveals a "partially" hyperbolic singularity topologically equivalent to

$$\frac{j}{2^{1/2}}\alpha\frac{\partial}{\partial\alpha} + \frac{(-1)^i a_i r^{i-2j+3}}{2^{1/2}\cdot 2^{i-2j+3}}\frac{\partial}{\partial r}.$$

Hence we have four cases to consider when the above jet is equivalent to the forms $\alpha(\partial/\partial\alpha) \pm r(\partial/\partial r)$ and $\alpha(\partial/\partial\alpha) \pm r^2(\partial/\partial r)$. Furthermore some of these cases subdivide depending on whether j is even or odd. We obtain the blow ups in Figs. 2–6.

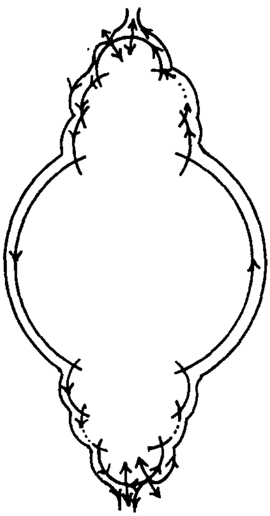


FIGURE 2

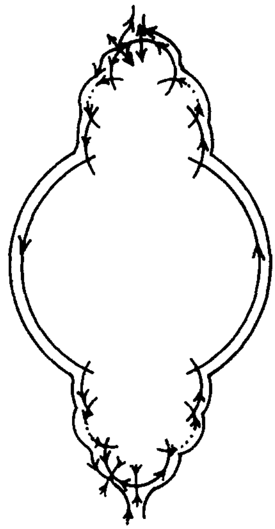


FIGURE 3

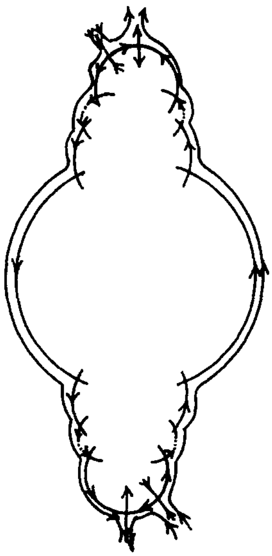


FIGURE 4

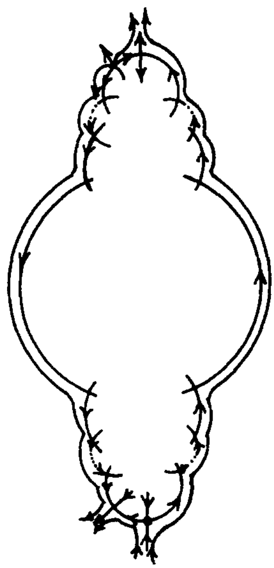


FIGURE 5

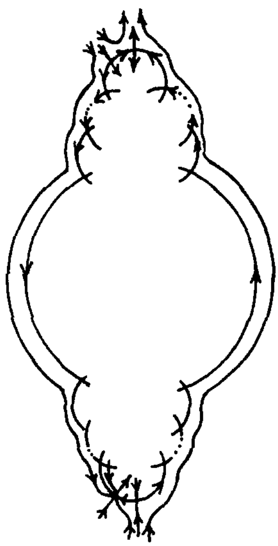


FIGURE 6

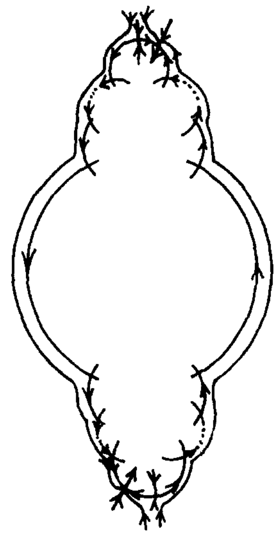


FIGURE 7

(b) $b_j = -1$

$$j_1(X_{j-1_{[0]}})(0) = j\alpha\frac{\partial}{\partial\alpha} - (j-1)r\frac{\partial}{\partial r}$$

$$j_1(X_{j-1_{[\pi/2]}})(0) = j\alpha\frac{\partial}{\partial\alpha} - r\frac{\partial}{\partial r}$$

$$j_1(X_{j-1_{[\pi/4]}})(0) = -j/2^{1/2}\alpha\frac{\partial}{\partial\alpha}.$$

The blow ups obtained here add only one new type of singularity namely Fig. 7.
The figures can be classified as follows.

$$\text{Fig. } 2: b_j = +1,\ \alpha\frac{\partial}{\partial\alpha} + r\frac{\partial}{\partial r},\ j \text{ odd}$$

$$\text{Fig. } 3: b_j = +1,\ \alpha\frac{\partial}{\partial\alpha} + r\frac{\partial}{\partial r},\ j \text{ even}$$

$$\text{Fig. } 4: b_j = +1,\ \alpha\frac{\partial}{\partial\alpha} - r\frac{\partial}{\partial r}$$

$$\text{Fig. } 5: b_j = +1,\ \alpha\frac{\partial}{\partial\alpha} + r^2\frac{\partial}{\partial r}$$

$$b_j = +1,\ \alpha\frac{\partial}{\partial\alpha} - r^2\frac{\partial}{\partial r},\ j \text{ odd}$$

$$\text{Fig. 6}: b_j = +1,\ \alpha \frac{\partial}{\partial \alpha} - r^2 \frac{\partial}{\partial r},\ j \text{ even}$$

$$b_j = -1,\ \alpha \frac{\partial}{\partial \alpha} + r^2 \frac{\partial}{\partial r}$$

$$b_j = -1,\ \alpha \frac{\partial}{\partial \alpha} - r^2 \frac{\partial}{\partial r}$$

$$\text{Fig. 7}: b_j = -1,\ \alpha \frac{\partial}{\partial \alpha} - r \frac{\partial}{\partial r}.$$

II. $W'_{4,3n} \backslash W'_{4,3n+1}$

$$= \{X \in W'_4 \mid a_2 = \cdots = a_{2n} = 0,\ a_{2n+1} \neq 0,\ b_2 = \cdots = b_n = 0,$$

$$\times\ b_{n+1}^2 + \frac{4}{n+1} a_{2n+1} \neq 0\} \cup \left\{\bigcup_{j=2}^{n} T_{3n+2-j,j}\right\}$$

$$\Big\{X \in W'_4 \mid a_2 = \cdots = a_{2n} = 0,\ a_{2n+1} \neq 0,\ b_2 = \cdots = b_n = 0,$$

$$\times\ b_{n+1}^2 \frac{4}{n+1} a_{2n+1} \neq 0\Big\}.$$

As in $I(i)$ blowing up X in the y-direction we obtain

$$X_1 = X_{[\pi/2]} = \left(xy + \sum_{k \geqslant n+1} b_k y^k\right) \frac{\partial}{\partial y} + \left(\sum_{k \geqslant 2n+1} a_k y^{k-1}\right.$$
$$\left. - x\left(x + \sum_{k \geqslant n+1} b_k y^{k-1}\right)\right) \frac{\partial}{\partial x}.$$

A polar blowing up of X_1 in its origin reveals singularities in $\alpha = 0$, (π) and $\alpha = \pi/2(3\pi/2)$ such that

$$j_1(X_{1_{[0]}}(0)) = 2\alpha \frac{\partial}{\partial \alpha} - r \frac{\partial}{\partial r}$$
$$j_1(X_{1_{[\pi/2]}}(0)) = 0.$$

By blowing up X_1 successively $(n - 1)$-times in the y-direction we obtain the vectorfield X_n where

$$j_2(X_n)(0) = (xy + b_{n+1} y^2)\, \partial/\partial y + (a_{2n+1} y^2 - nx^2 - nb_{n+1} xy)\, \partial/\partial x.$$

The polar blow-up of X_n, $\bar{X}_n$ is such that

$$\bar{X}_{n_0} = \bar{X}_n \mid S^1 \times \{0\} = ((n + 1) \cos^2\alpha \sin \alpha + (n + 1) b_{n+1} \cos \alpha \sin^2\alpha$$
$$- a_{2n+1} \sin^3\alpha) \frac{\partial}{\partial \alpha}.$$

Therefore $\bar{X}_{n_0}$ has singularities in $\sin\alpha_0 = 0$, $\cot\alpha_1 = -\frac{1}{2}b_{n+1} + \frac{1}{2}(b_{n+1} + [4/(n+1)]\,a_{2n+1})^{1/2}$ $(=A)$ and $\cot\alpha_2 = -\frac{1}{2}b_{n+1} - \frac{1}{2}(b_{n+1}^2 + [4/(n+1)]\,a_{2n+1})^{1/2}$ $(=B)$.

We have to consider the following cases.

$$\text{(i)}\quad b_{n+1}^2 + \frac{4}{n+1}\,a_{2n+1} < 0$$

In this case the only singularity of $\bar{X}_{n_0}$ is at $\alpha_0 = 0$, (π) and $j_1(X_{n_{[0]}})(0) = (n+1)\,\alpha(\partial/\partial\alpha) - nr(\partial/\partial r)$.

It can easily be seen that in this case the vector fields do not have any separatrices and hence are contained in $\bar{R}$.

$$\text{(ii)}\quad b_{n+1}^2 + \frac{4}{n+1}\,a_{2n+1} > 0$$

X_n has the same singularity in $\alpha_0 = 0$ as in (a). The radial eigenvalues of X_n in α_1 and α_2 are found by tedious but straightforward calculations to be $(-B + [a_{2n+1}/(n+1)]A)/D^3$ and $(-A + [a_{2n+1}/(n+1)]B)/D^3$ respectively where $D = (b_{n+1}^2 + [4/(n+1)]\,a_{2n+1})^{1/2}$. It follows that if we take $j_1(X_{n_{[\alpha_i]}})(0) = a_i\alpha(\partial/\partial\alpha) + b_i r(\partial/\partial r)$ for $i = 0, 1, 2$ we have the cases

$$a_{2n+1} > 0 \begin{cases} b_{n+1} > 0 - a_1 < 0, & b_1 > 0; \quad a_2 > 0, \quad b_2 < 0 \\ b_{n+1} = 0 - a_1 < 0, & b_1 > 0; \quad a_2 > 0, \quad b_2 < 0 \\ b_{n+1} < 0 - a_1 < 0, & b_1 > 0; \quad a_2 > 0, \quad b_2 < 0 \end{cases}$$

$$a_{2n+1} < 0 \begin{cases} b_{n+1} > 0 - a_1 < 0,\, b_1 > 0;\, a_2 > 0,\, b_2 > 0 \\ b_{n+1} < 0 - a_1 < 0,\, b_1 < 0;\, a_2 > 0,\, b_2 < 0 \end{cases}$$

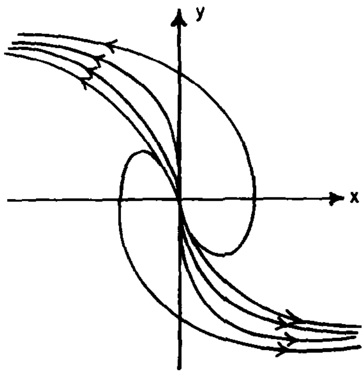
We find only four topologically distinct cases which correspond to vector fields of the following type:

Fig. 2(a): $a_{2n+1} < 0$, $b_{n+1} > 0$, n even.
Fig. 3(a): $a_{2n+1} < 0$, n odd.
Fig. 4(a): $a_{2n+1} > 0$.
Fig. 7(a): $a_{2n+1} < 0$, $b_{n+1} < 0$, n even.

III. $W'_{4,3n+1} \backslash W'_{4,3n+2}$

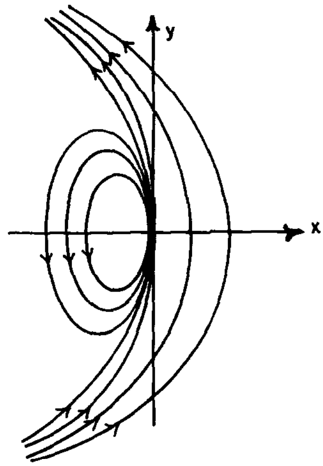
$$= \Big\{X \in W'_4 \mid a_2 = \cdots = a_{2n} = 0,\, a_{2n+1} \neq 0,\, b_2 \cdots = b_n = 0,$$

$$\times\, b_{n+1}^2 + \frac{4}{n+1}\,a_{2n+1} = 0\Big\} \cup \left\{\bigcup_{j=2}^{n+1} T_{3n+3-j,j}\right\}$$



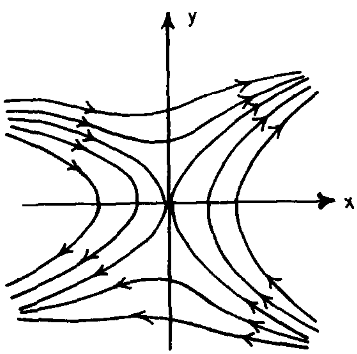
$X(x, y) = (x + y^3)\partial/\partial y - y^7\,\partial/\partial x.$

FIG. 2a



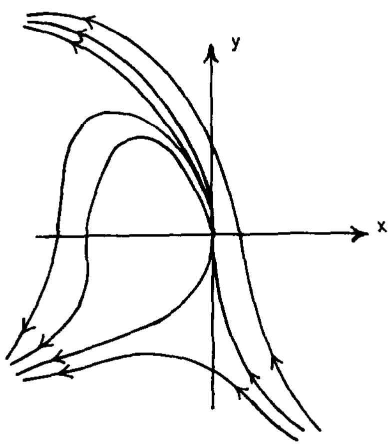
$X(x, y) = (x + y^2)\partial/\partial y - y^5\,\partial/\partial x.$

FIG. 3a



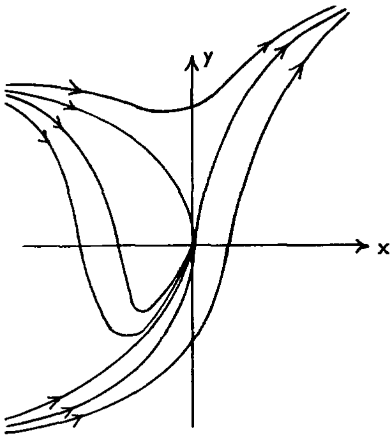
$X(x, y) = (x + y^2)\,\partial/\partial y + y^5\,\partial/\partial x.$

FIG. 4a



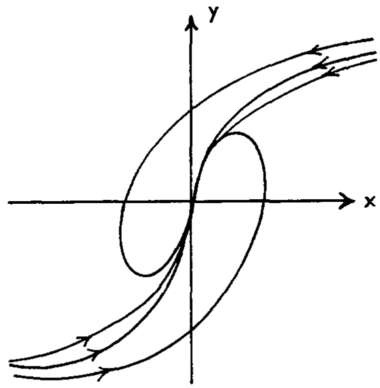
$X(x, y) = (x + y^2)\,\partial/\partial y - y^6\,\partial/\partial x.$

FIG. 5a



$X(x, y) = (x + y^2)\partial/\partial y + y^4\,\partial/\partial x.$

FIG. 6a



$X(x, y) = (x - y^3)/\partial y - y^7\,\partial/\partial x.$

FIG. 7a

Using II, the vector field

$$\bar{X}_{n_0} = ((n+1)\cos^2\alpha\sin\alpha + (n+1)b_{n+1}\cos\alpha\sin^2\alpha - a_{2n+1}\sin^3\alpha)\frac{\partial}{\partial\alpha}$$

and has singularities at $\alpha = 0$, (π) and at $\cot\alpha_1 = -\frac{1}{2}b_{n+1}$. As before $j_1(X_{n_{[0]}})(0) = (n+1)\,\alpha(\partial/\partial\alpha) - nr(\partial/\partial r)$. The singularity $\alpha = \alpha_1$ is partially hyperbolic of topological type

$$(n+1)(b_{n+1}^2 + 4)^2\alpha^2\frac{\partial}{\partial\alpha} + b_{n+1}(b_{n+1}^2 + 4)r\frac{\partial}{\partial r}$$

For this case it is not difficult to obtain singularity schemes giving vector fields topologically equivalent to:

Fig. 2(a): $b_{n+1} > 0$, n even.
Fig. 3(a): $b_{n+1} \neq 0$, n odd.
Fig. 7(a): $b_{n+1} < 0$, n even.

Remark. It is straightforward to see that each of the classified subsets of W_4 which we have obtained is C^0-stable.

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